RAYNAUD-MUKAI CONSTRUCTION AND CALABI-YAU THREEFOLDS IN POSITIVE CHARACTERISTIC

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Abstract. In this article, we study the possibility of producing a Calabi-Yau threefold in positive characteristic which is a counter-example to Kodaira vanishing. The only known method to construct the counter-example is so called inductive method such as the Raynaud-Mukai construction or Russel construction. We consider Mukai's method and its modification. Finally, as an application of Shepherd-Barron vanishing theorem of Fano threefolds, we compute $H^1(X, H^{-1})$ for any ample line bundle H on a Calabi-Yau threefold X on which Kodaira vanishing fails.

1. Introduction

Although every K3 surface in positive characteristic can be lifted to characteristic 0 [2], there are some non-liftable Calabi-Yau threefolds, namely a smooth threefold X with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. If a Calabi-Yau polarized threefold (X, L) over the field k of $\mathrm{char}(k) = p \geq 3$ is a counter-example to Kodaira vanishing, i.e., $H^i(X, L^{-1}) \neq 0$ for $i = 1$ or $i = 2$, X is non-liftable to the second Witt vector ring $W_2(K)$ (and the Witt vector ring $W(k)$) by the cerebrated Raynaud-Deligne-Illusies version of Kodaira vanishing theorem [3]. But this does not necessarily imply that X cannot be liftable to characteristic 0. Moreover, a non-liftable variety is not necessarily a counter-example to Kodaira vanishing and as far as the author is aware, it is not known whether Kodaira vanishing holds for the non-liftable Calabi-Yau threefolds [6, 7, 8, 16, 4, 1] that have been found so far. We do not even know whether Kodaira vanishing holds for all Calabi-Yau threefolds. Thus Kodaira type vanishing for Calabi-Yau threefolds is an interesting problem, which is independent from but seems to be closely related to the lifting problem.

A counter-example to Kodaira vanishing has been given by M. Raynaud, which is a surface over a curve [14]. This example was extended to arbitrary dimension by S. Mukai [11, 12], which we will call the Raynaud-Mukai construction or, simply, Mukai construction.

The idea is, so to say, an inductive construction. Namely, we start from a polarized smooth curve (C, D). The ample divisor D satisfies a special condition, which is a sufficient condition for the non-vanishing $H^1(X, \mathcal{O}_X(-D)) \neq 0$, and called a (pre-)Tango structure. Then we give an algorithm to construct from a variety X with a (pre-)Tango structure D a new variety $\tilde{X}$ with a higher dimensional (pre-)Tango structure $\tilde{D}$ such that $\dim \tilde{X} = \dim X + 1$, using cyclic cover technique. There is another way of constructing counter-examples using quotient of p-closed differential forms [15, 19]). But this is also an inductive construction and the obtained varieties

are the same as the Raynaud-Mukai construction [19]. As far as the author is aware, non-inductive construction of higher dimensional counter-examples is not yet found.

In this paper, we consider the problem of whether we can construct a Calabi-Yau threefold with Kodaira non-vanishing by Mukai construction or by its modification. Section 2 presents the Raynaud-Mukai construction. For $p \geq 5$, Raynaud-Mukai varieties are of general type so that the only possibility resides in the cases of $p = 2, 3$. Then in section 3, we will see that Mukai construction does not produce any K3 surfaces or Calabi-Yau threefolds (Corollary 9 and Corollary 10). Then we consider possible modifications of the Raynaud-Mukai construction: we keep the inductive construction but give up obtaining a (pre-)Tango structure. We show that if there exists a surface X of general type together with a (pre-)Tango structure D satisfying some property (this is not obtained by Mukai construction), we can construct a Calabi-Yau threefold $\tilde{X}$ with a (pre-)Tango structure $\tilde{D}$ (Corollary 11) and describe the cohomology $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ in certain situations (Proposition 13). Unfortunately, we could not prove or disprove existence of such a polarized surface (X, D).

Finally, in section 3 we show that if Kodaira non-vanishing $H^1(X, L^{-1}) \neq 0$ holds for a polarized Calabi-Yau threefold (X, L) over the field k of $\operatorname{char} k = p \geq 5$ satisfying the condition that L^ℓ is a Tango-structure for some $\ell \geq 1$, we compute the cohomology $H^1(X, H^{-1})$ for any ample line bundle H of X (Theorem 18, Corollary 19).

2. The Raynaud-Mukai construction

In this section, we present the Raynaud-Mukai construction. Although [12] is available now, we prefer to use the version described in [11], which is slightly different from the 2005 version. As 1979 version is only available in Japanese, we present some details for the readers convenience.

The idea is to construct from a counter-example to Kodaira vanishing, i.e., a polarized variety (X, L) with $H^1(X, L^{-1}) \neq 0$ a new counter-example $(\tilde{X}, \tilde{L})$ with $\dim \tilde{X} = \dim X + 1$. This inductive construction starts from a polarized curve (X, L) called a Tango-Raynaud curve.

2.1. pre-Tango structure and Kodaira non-vanishing.

Definition 1 (pre-Tango structure)**.** *Let X be a smooth projective variety. Then an ample divisor D, or an ample line bundle $L = \mathcal{O}_X(D)$, is called a* pre-Tango structure *if there exists an element $\eta \in k(X) \backslash k(X)^p$, where $k(X)$ denotes the function field of X, such that the Kähler differential is $d\eta \in \Omega_X(-pD)$, which will be simply denoted as $(d\eta) \geq pD$. In this paper, the element η will be called a* justification *of the pre-Tango structure.*

Existence of a pre-Tango structure implies Kodaira non-vanishing. In fact, consider the absolute Frobenius morphism

$$F : \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(-pD)$$

such that $F(a) = a^p$ for $a \in \mathcal{O}_X$ and set $B_X(-D) := \operatorname{Coker} F$. Then we have

$$0 \longrightarrow H^0(X, B_X(-D)) \longrightarrow H^1(X, \mathcal{O}_X(-D)) \xrightarrow{F} H^1(X, \mathcal{O}_X(-pD))$$

and then we can show

Proposition 2. *$H^0(X, B_X(-D)) = \{df \in k(X) \mid (df) \geq pD\}$.*

Thus, if there exists a pre-Tango structure D and $\dim X \geq 2$, then we have Kodaira non-vanishing: $H^1(X, \mathcal{O}_X(-D)) \neq 0$.

Notice that the inclusion $H^0(X, B_X(-D)) \subset H^1(X, \mathcal{O}_X(-D))$ may be strict, so that there is a possibility that a non pre-Tango structure L causes a Kodaira non-vanishing. However, since the iterated Frobenius map

$$F^e : H^1(X, L^{-1}) \longrightarrow H^1(X, L^{-p^e})$$

is trivial for $e \gg 0$, L^n is a pre-Tango structure for sufficiently large $n \in \mathbb{N}$.

Pre-Tango structure for curves are characterized by the Tango-invariant [21, 20]. Let C be a smooth projective curve of genus g (≥ 2). Then the Tango-invariant is defined as

$$n(C) = \max\left\{\deg\left[\frac{df}{p}\right] \;:\; f \in k(X)/k(X)^p\right\}$$

where $[\cdots]$ denotes the round up. We easily know that

$$0 \leq n(C) \leq \frac{2(g-1)}{p}.$$

Then, C has a pre-Tango structure D if $n(C) > 0$. We just set $D = \left[\frac{(df)}{p}\right]$ and then D is ample on C such that $(df) \geq pD$.

In the following, we will call the pair (X, L) in Definition 1 a *pre-Tango polarization*. The Raynaud-Mukai construction is an algorithm to make a new pre-Tango polarization from a pre-Tango polarization whose dimension is lower by one.

2.2. **purely inseparable cover.**

From a pre-Tango polarized variety (X, L) we can construct a reduced and irreducible purely inseparable cover $\tau : G \longrightarrow X$ of degree p. Conversely, existence of such a cover implies existence of a pre-Tango polarization.

2.2.1. *Construction and characterization.* Given a pre-Tango polarized variety $(X, L = \mathcal{O}_X(D))$, choose an element $(0 \neq)\eta \in H^0(X, B_X(-D))(= \operatorname{Ker} F)$. Then we have a corresponding non-split short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \longrightarrow 0 \tag{1}$$

where E is a rank 2 vector bundle on X. Taking the Frobenius pull-back, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E^{(p)} \longrightarrow L^{(p)} \longrightarrow 0.$$

where, for example, $E^{(p)} = E \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ with $F : \mathcal{O}_X \longrightarrow \mathcal{O}_{X'}$ the Frobenius morphism. Notice that the new sequence corresponds to $F(\eta) = 0$ so that it splits and by using the split maps, we obtain the sequence with the reverse arrows

$$0 \longleftarrow \mathcal{O}_X \longleftarrow E^{(p)} \longleftarrow L^{(p)} \longleftarrow 0.$$

Tensoring by $L^{(p)^{-1}}$ over $\mathcal{O}_X$, we finally obtain the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E^{(p)} \otimes L^{(p)^{-1}} \longrightarrow L^{(p)^{-1}} \longrightarrow 0. \tag{2}$$

Now we consider the $\mathbb{P}^1$-fibration

$$\pi : P = \mathbb{P}(E) \longrightarrow X$$

together with the canonical section $F \subset P$, which is defined by the image of $1 \in \mathcal{O}_X$ in E, and

$$\pi^{(p)} : P^{(p)} = \mathbb{P}(E^{(p)} \otimes L^{(p)^{-1}}) \cong \mathbb{P}(E^{(p)}) \longrightarrow X$$

together with the canonical section $F^{(p)} \subset P^{(p)}$ which is the image of $1 \in \mathcal{O}_X$ in $E^{(p)} \otimes L^{(p)^{-1}}$, corresponding to (1) and (2). Moreover, we consider the relative Frobenius morphism $\psi : P \longrightarrow P^{(p)}$ over X. On an open set $U \subset X$ such that $E|_U \cong \mathcal{O}_U^r$ with $r = \operatorname{rank} E$, ψ is induced by the local morphism $E^{(p)}|_U \cong \mathcal{O}_U^r \otimes_{\mathcal{O}_U} \mathcal{O}_{U'} \to E|_U$ sending $\sum_{i=1}^r a_i \otimes f = \sum_i 1 \otimes a_i^p f \in \mathcal{O}_U^r$ to $\sum_i a_i^p f \in \mathcal{O}_U^r$. Thus, on a fiber $\pi^{-1}(x) \cong \mathbb{P}^1$, $\psi|_{\pi^{-1}(x)} : \pi^{-1}(x) \longrightarrow \pi^{(p)^{-1}}(x)$ is the Frobenius pull-back, i.e., $\psi(a,b) = (a^p, b^p)$ for every projective coordinate $(a,b) \in \pi^{-1}(x)$. Now consider the scheme theoretic inverse image of $F^{(p)}$ inside P:

$$G := \psi^{-1}(F^{(p)}) \subset P$$

Then we can show

Proposition 3. *(1) $G \cap F = \emptyset$,*
(2) $\mathcal{O}_P(G) \cong \mathcal{O}_P(p) \otimes \pi^* L^{-p} \cong \mathcal{O}_P(pF - p\pi^* D)$, and
(3) $\rho = \pi|_G : G \longrightarrow X$ is a purely inseparable cover of degree p.

We can show that existence of such a G characterizes pre-Tango structure. To summarize, we have

Theorem 4 (See Proposition 1.1 in [12])**.** *Let X be a smooth projective variety of characteristic $p > 0$ and L be an ample line bundle. Then the following are equivalent:*

(1) *L is a pre-Tango structure.*
(2) *There exists a $\mathbb{P}^1$-bundle $\pi : P \longrightarrow X$ and a reduced irreducible effective divisor $G \subset P$ such that*
   (a) *$\rho : G \longrightarrow X$ is a purely inseparable cover of degree p*
   (b) *$P = \mathbb{P}(E)$ where E is a rank 2 vector bundle on X such that*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \longrightarrow 0$$

2.2.2. *smoothness.* For smoothness of the purely inseparable cover G, we have

Theorem 5 (S. Mukai [12])**.** *Let (X, D) be a pre-Tango polarized variety over the field of characteristic $p > 0$ and G is the purely inseparable cover constructed from a justification $(0 \neq)\eta \in k(X) \backslash k(X)^p$. Then G is smooth if and only if $(d\eta) = pD$. This means that for the multiplication by $d\eta$*

$$\mathcal{O}_X(pD) \xrightarrow{d\eta} \Omega_X \longrightarrow \operatorname{Coker}(d\eta),$$

$\operatorname{Coker} d\eta$ is locally free at every $x \in X$.

Proof. For a proof in the case of $\dim X = 2$, see Theorem 3 [18]. ∎

Definition 6 (Tango structure)**.** *Let X be a smooth projective variety with a pre-Tango structure $L = \mathcal{O}_X(D)$. Then, D, or L, is called a* Tango structure *if and only if a justification $\eta \in k(X)\backslash k(X)^p$ satisfies $(d\eta) = pD$. In this case, the pre-Tango polarization (X, L) or (X, D) will be called a* Tango polarization.

A smooth projective curve X of genus $g \geq 2$ with a Tango structure D is called a *Tango-Raynaud curve*. For examples of Tango-Raynaud curves, see for example [14, 11, 12].

2.3. **cyclic cover.** Let (X, D) be a pre-Tango polarization and D is divided by $k \in \mathbb{N}$ with $(p, k) = 1$ and we have $D = kD'$. If X is a curve, we can divide D by any natural number k dividing $\deg D$ using the theory of Jacobian variety (cf. page 62 of [13]). But the condition $(p, k) = 1$ is necessary for the covering to be cyclic.

Now we construct a kth cyclic cover of the $\mathbb{P}^1$-fibration $\pi : P \longrightarrow X$ ramified over $F + G$, which means that π is ramified at the reduced preimage of $F + G$. There are at least two well-known constructions.

The first one is rather explicit and is suitable for computing cohomologies (cf. [18]). We first choose $m \in \mathbb{N}$ such that $k|(p + m)$ and set $\mathcal{M} = \mathcal{O}_P\left(-\frac{p+m}{k}F\right) \otimes \pi^*\mathcal{O}_X(pD')$. Then we have $\mathcal{M}^{\otimes k} = \mathcal{O}_P(-mF) \otimes \mathcal{O}_P(-pF) \otimes \pi^*\mathcal{O}_X(pD) = \mathcal{O}_P(-mF - G)$ by Proposition 3. Then we can introduce $\bigoplus_{i=0}^{k-1} \mathcal{M}^{\otimes i}$ the structure of a graded $\mathcal{O}_P$-algebra by defining multiplication $\mathcal{M}^{\otimes i} \times \mathcal{M}^{\otimes j} \longrightarrow \mathcal{M}^{\otimes i+j}$ s. t. $(a.b) \mapsto a \otimes b$ if $i+j < k$ and $\mathcal{M}^{\otimes i} \times \mathcal{M}^{\otimes j} \longrightarrow \mathcal{M}^{\otimes i+j} \longrightarrow \mathcal{M}^{\otimes i+j-k}$ s. t. $(a.b) \mapsto a \otimes b \mapsto a \otimes b \otimes \xi$ if $i+j \geq k$ where we choose a non-trivial element $\xi \in \mathcal{O}_P(mF+G)$ such that $mF+G$ is the zero locus of ξ. Now we consider the affine morphism $X' := \mathcal{S}pec\ \bigoplus_{i=0}^{k-1} \mathcal{M}^{\otimes i} \to P$ and this is the cyclic cover ramified over $mF + G$. Since X is smooth, $F \cong X$ is also smooth. Moreover if D is a Tango structure and G is smooth by Theorem 5, then X' is smooth if and only if $m = 1$; if $m > 1$ then X' is singular along F, which may cause non-normality of X'. Normalization of X', if necessary, is carried out by Esnault-Viehwegs method (see § 3 of [5]). $\tilde{X} = \mathcal{S}pec \bigoplus_{i=0}^{k-1} M^{\otimes i} \otimes \mathcal{O}_P\left(\left[\frac{i(mF+G)}{k}\right]\right)$ and this is smooth if D is Tango. We note that this normalization procedure highly depends on the condition $(p, k) = 1$ since we use the kth root of unity. Then we set the natural morphism $\varphi : \tilde{X} \longrightarrow X' \longrightarrow P$.

The second construction uses normalization. Since we have liner equivalence $G \sim pF - p\pi^*(D)$ there exist a function $R \in k(P)$ such that $(R) = G - (pF - p\pi^*(D)) = G - (pF - pk\pi^*(D'))$. Then let $\tilde{X}$ be the normalization of P in the finite extension $k(P)(R^{1/k})$ of $k(P)$ and $\varphi : \tilde{X} \longrightarrow P$ be the normalization morphism. Then we set $f = \pi \circ \varphi$. Now if we work locally we know that there exist divisors $\tilde{G}$ and $\tilde{F}$ on $\tilde{X}$ such that $\varphi^*F = k\tilde{F}$ and $\varphi^*G = k\tilde{G}$. Moreover, we have $\tilde{G} \sim p\tilde{F} - pf^*(D')$ on $\tilde{X}$. We note that the condition $(p, k) = 1$ is necessary to assure the existence of $\tilde{F}$, division of F by k. Otherwise, if $k = p^\ell r$ with $\ell \geq 1$ and $(p, r) = 1$ we have $\tilde{F} \subset \tilde{X}$ such that $\varphi^*F = k'\tilde{F}$ with $k' = p^{\ell-1}r = k/p$. $\tilde{X}$ is smooth if D is Tango.

Now we set $f := \pi \circ \varphi : \tilde{X} \longrightarrow X$, which is actually a fibration of rational curves with moving singularities, i.e., rational curves with cusp singularity of type $x^p = y^t$ at $\tilde{G}$.

2.4. **polarization.** The cyclic cover $\tilde{X}$ of the $\mathbb{P}^1$-fibration is a counter-example to Kodaira vanishing because the polarization $\tilde{D} = (k-1)\tilde{F} + f^*(D')$ causes non-vanishing $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{D})) \neq 0$. In fact, $\tilde{D}$ is ample (see Sublemma 1.6 [12]) and we have

Proposition 7. *Suppose $\tilde{X}$ is as above, then $\tilde{D}$ is a Tango structure of $\tilde{X}$ and in particular we have Kodaira non-vanishing $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{D})) \neq 0$.*

This result is stated in [11] without proof and in the case of $k \equiv 1 \bmod p$ a proof using Maruyama's elementary transformation [10] is given in [12]. We give here a proof of general case.

Proof. Let $\tilde{\eta} = R^{1/k} \in k(\tilde{X})$. Since $(\tilde{\eta}) = \tilde{G} - p\tilde{F} + pf^*(D')$, $\tilde{\eta}$ is locally described as $\tilde{\eta} = g(\delta\phi^{-1})^p$ where g, ϕ and δ are local equations defining $\tilde{G}$, $\tilde{F}$ and $f^*(D')$. Then its Kähler differential is

$$d\tilde{\eta} = (\delta\phi^{-1})^p dg = (\phi^{k-1}\delta)^p \phi^{-pk} dg. \tag{3}$$

Now we consider dg. As a Cartier divisor we describe $D = \{(U_i, g_i)\}_i$ for an open cover $X = \bigcup_i U_i$ and $g_i \in k(X)$. Since D is a pre-Tango structure, there exists a justification $\eta \in k(X)$ such that $(d\eta) \geq pD$, which locally means that we have $\eta|_{U_i} = g_i^p c_i$ for some $c_i \in \mathcal{O}_{U_i}$ so that we have $(d\eta)|_{U_i} = g_i^p dc_i$. Then, as in Proposition 1 [18], $G \subset P$ is locally described as

$$\operatorname{Proj} \mathcal{O}_{U_i}[x, y]/(c_i x^p + y^p)$$

where x is the (local) coordinate corresponding to the canonical section F of $\pi : P \longrightarrow X$. Hence the local defining equation of $G \subset P$ is $c_i x^p + y^p$, and since $\varphi^* F = k\tilde{F}$ and $\varphi^* G = k\tilde{G}$, the defining equation of $\tilde{G}$ is $g = c_i Z^{kp} + W^{kp}$, where Z is the local coordinate of $\tilde{X}$ corresponding to $\tilde{F}$, namely $Z = \varepsilon\phi$ with some local unit ε. Thus we have

$$dg = \varepsilon^{pk}\phi^{pk} dc_i. \tag{4}$$

Thus by (3) and (4) we obtain $d\tilde{\eta} = (\delta\phi^{-1})^p dg = \varepsilon^{pk}(\phi^{k-1}\delta)^p dc_i$ so that

$$(d\tilde{\eta}) \geq p((k-1)\tilde{F} + f^*D') = p\tilde{D}$$

where the equality holds if $(d\eta) = pD$, i.e., if D is a Tango structure. $\square$

3. Calabi-Yau threefolds and the Raynaud-Mukai construction

3.1. **Raynaud-Mukai varieties cannot be Calabi-Yau.** The aim of this section is to show that Mukai construction does not produce K3 surfaces or Calabi-Yau threefolds. Notice that Raynaud-Mukai variety is always of general type for $p \geq 5$ (cf. Prop. 7 [11] or Prop. 2.6 [12]) so that the only possibility is the case $p = 2, 3$.

Now let (X, D), $D = kD'$ ($k \in \mathbb{N}$), $\pi : P \longrightarrow X$, $F, G \subset P$, $(\tilde{X}, \tilde{D})$, $\varphi : \tilde{X} \longrightarrow P$ and $f : \tilde{X} \longrightarrow X$ be as in the previous section. The canonical divisor of X will be simply denoted by K. Now we have

Proposition 8 (cf. Prop. 7 [11]). *Let $\tilde{K}$ be the canonical divisor of $\tilde{X}$. Then we have*

$$\tilde{K} \sim (pk - p - k - 1)\tilde{F} + f^*(K - (pk - p - k)D')$$

Proof. Since the finite morphism $\varphi : \tilde{X} \longrightarrow P$ is ramified at $\tilde{F} = (\varphi^*(F))_{red}$ and $\tilde{G} = (\varphi^*(G))_{red}$ with the same ramification index k and $F + G \sim (p+1)F - pk\pi^*D'$, we compute

$$\begin{aligned} \tilde{K} &\sim \varphi^*K_P + (k-1)(\tilde{F} + \tilde{G}) \quad \text{by ramification formula} \\ &\sim \varphi^*K_P + (k-1)\frac{1}{k}\varphi^*(F+G) \\ &\sim \varphi^*K_p + (k-1)((p+1)\tilde{F} - pf^*D'). \end{aligned}$$

Moreover, since E is the rank 2 vector bundle satisfying

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{O}_X(kD') \longrightarrow 0,$$

we have $K_P \sim -2F + \pi^*(K + kD')$. Then we obtain the required formula. $\square$

We notice that since $\operatorname{Pic} P \cong \mathbb{Z} \cdot [F] \oplus \pi^* \operatorname{Pic} X$ and φ is finite, we have $\operatorname{Pic} \tilde{X} \cong \mathbb{Z} \cdot [\tilde{F}] \oplus f^* \operatorname{Pic} X$. This fact will be used implicitly in the following discussion.

Corollary 9. *A Raynaud-Mukai surface can never be a K3 surface.*

Proof. Assuming $\dim \tilde{X} = 2$, we have only to show that we never have $\tilde{K} \sim 0$. Assume that we have $\tilde{K} \sim (pk - p - k - 1)\tilde{F} + f^*(K - (pk - p - k)D') \sim 0$, from which have two relations $pk - p - k - 1 = 0$ and $K - (pk - p - k)D' = 0$. By the first relation, we have $k = \frac{p+1}{p-1} \in \mathbb{N}$, so that we must have $p = 2$ and $k = 3$ or $p = 3$ and $k = 2$. This implies $K = D'$ by the second relation. However, since (X, D) is a (pre-)Tango polarized curve, we have $(d\eta) \geq pD$ for some justification $\eta \in k(X)$, namely $D' = K \geq pD = pkD'$, which is impossible unless $pk = 1$. $\square$

By a similar discussion to the proof of Corollary 9, we can also show

Corollary 10. *A Raynaud-Mukai threefold can never be Calabi-Yau.*

Proof. Let $\tilde{X}$ be a Mukai threefold obtained from a Mukai surface X with a (pre-)Tango structure $D = kD'$ as a kth cyclic cover of the $\mathbb{P}^1$-fibration P and assume that $\tilde{K} \sim 0$. Then as in the proof of Corollary 9 we have $(p, k) = (2, 3)$ or $(3, 2)$ and

$$K \sim D'. \tag{5}$$

Now we will consider the situation whose dimensions are all lower by one. Namely, let the surface X be constructed from a (pre-)Tango polarized curve (X_1, D_1) with $D_1 = k_1 D_1'$. We have the k_1th cyclic cover $\varphi_1 : X \longrightarrow P_1$ of the $\mathbb{P}^1$-fibering $\pi_1 : P_1 \longrightarrow X_1$ ramified over $F_1 + G_1$ and $\tilde{F}_1 = (\varphi_1^*(F_1))_{red}$ and $\tilde{G}_1 = (\varphi_1^*(G_1))_{red}$

have the same ramified index k_1. We set $f_1 = \pi_1 \circ \varphi_1$. Then by Proposition 8, we have

$$K \sim (pk_1 - p - k_1 - 1)\tilde{F}_1 + f_1^*(K_1 - (pk_1 - p - k_1)D_1')$$

Since we have $(kD' =)D = (k_1 - 1)\tilde{F}_1 + f_1^*(D_1')$ by definition, the condition (5) entails

$$\left(pk_1 - p - k_1 - 1 - \frac{k_1 - 1}{k}\right)\tilde{F}_1 + f_1^*\left(K_1 - (pk_1 - p - k_1 + \frac{1}{k})D_1'\right) \sim 0.$$

Then the coefficient of $\tilde{F}_1$ must be 0 so that we have

$$k_1 = \frac{k(p+1) - 1}{k(p-1) - 1} = \begin{cases} 4 & \text{if } p = 2 \\ \\ \dfrac{7}{3} & \text{if } p = 3 \end{cases}$$

But since we must have $k_1 \in \mathbb{N}$ and $(k_1, p) = 1$, these values of k_1 are not allowed. $\square$

3.2. **a modification of the Raynaud-Mukai construction.** The Raynaud-Mukai construction is an algorithm to construct from a given (pre-)Tango polarization (X, D) with $D = kD'$ a new (pre-)Tango polarization $(\tilde{X}, \tilde{D})$ with $\dim X = \dim \tilde{X} - 1$ by taking a kth cyclic cover. We apply this procedure inductively starting from a (pre-)Tango polarized curve. We have seen in the previous subsection that the essential reason that the Raynaud-Mukai construction does not produce Calabi-Yau threefolds is that we cannot find the degree k cyclic covers with $(p, k) = 1$ in all inductive steps.

Now we will consider some modification of the Raynaud-Mukai construction. There are following two possibilities.

(I) Let (X, D) be a (pre-)Tango polarized surface obtained by a method other than Mukai construction. Then apply the Raynaud-Mukai construction to obtain a (pre-)Tango polarized threefold $(\tilde{X}, \tilde{D})$.

(II) Let (X, D) be a (pre-)Tango polarized surface by the Raynaud-Mukai construction. Then we construct a Calabi-Yau threefold in a similar way to Mukai construction. Namely, we do not assume the condition $(p, k) = 1$ for the degree k of "cyclic cover ".

The Calabi-Yau threefolds obtained by (I) are counter-examples to Kodaira vanishing. The surface X required in (I) is precisely as follows:

Corollary 11. *Let (X, D) a (pre-)Tango polarized surface with $D = kD'$ for some $k \in \mathbb{N}$. Then the Raynaud-Mukai construction gives a polarized Calabi-Yau threefold $(\tilde{X}, \tilde{D})$ by a kth cyclic cover if and only if*

(i) $(p, k) = (2, 3)$ or $(3, 2)$, and

(ii) $D = kD'$ for some ample D' and $K_X \sim D'$.

In particular, X is a surface of general type.

Proof. By the same discussion as in the proof of Corollary 9 and 10. $\square$

Unfortunately we do not know how to construct a polarized surface (X, D) as in Corollary 11. But Theorem 12(i) below seems to indicate a possibility.

Theorem 12 (S. Mukai [11]). *Let X be a (smooth) surface over the field k of* $\operatorname{char} k = p > 0$. *Assume that Kodaira vanishing fails on X. Then we have*

(*i*) *X is of general type or quasi-elliptic surface with Kodaira dimension* 1 *(if $p = 2, 3$).*

(*ii*) *There exists a surface X' birationally equivalent with X such that there is a morphism $g : X' \longrightarrow C$ to a curve C whose fibers are all connected and singular.*

It is proved that, in the case of surfaces, Kodaira (non-)vanishing is preserved in birational equivalence (see Corollary 8 [20]). Thus by Theorem 12(*ii*) it seems to be reasonable to consider a fibration $\rho : X \longrightarrow C$ to a curve.

For a Calabi-Yau threefold, we often assume simple connectedness which implies $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for our example. For this property, we have the following.

Proposition 13. *Assume that the surface X in Corollary 11 has a fibration over a curve C: $g : X \longrightarrow C$ and set $h : \tilde{X} \xrightarrow{f} X \xrightarrow{g} C$. Then we have $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(C, g_*\mathcal{O}_X) \oplus H^0(C, R^1h_*\mathcal{O}_{\tilde{X}})$.*

Proof. Consider the Leray spectral sequence

$$E_2^{pq} = H^p(C, R^q h_*\mathcal{O}_{\tilde{X}}) \Rightarrow H^{p+q}(\tilde{X}, \mathcal{O}_{\tilde{X}}).$$

Then by the 5-term exact sequence we have

$$0 \longrightarrow H^1(C, h_*\mathcal{O}_{\tilde{X}}) \longrightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}). \longrightarrow H^0(C, R^1h_*\mathcal{O}_{\tilde{X}}) \longrightarrow H^2(C, h_*\mathcal{O}_{\tilde{X}})$$

where the last term $H^2(C, h_*\mathcal{O}_{\tilde{X}})$ vanishes since $\dim C < 2$. Thus we have

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^1(C, h_*\mathcal{O}_{\tilde{X}}) \oplus H^0(C, R^1h_*\mathcal{O}_{\tilde{X}}).$$

On the other hand, we have $(p, k) = (2, 3)$ or $(3, 2)$ by Corollary 11 and the explicit construction of the cyclic cover gives

$$\tilde{X} = \begin{cases} \mathcal{S}pec \bigoplus_{i=0}^{2} \mathcal{O}_P(-i) \otimes \pi^*(2iD') & \text{if } (p,k) = (2,3) \\ \mathcal{S}pec \bigoplus_{i=0}^{1} \mathcal{O}_P(-2i) \otimes \pi^*(3iD') & \text{if } (p,k) = (3,2) \end{cases}$$

where $\pi : P \longrightarrow X$ is the $\mathbb{P}^1$-fibering. Thus we compute

$$\begin{aligned} h_*\mathcal{O}_{\tilde{X}} &= (g \circ \pi \circ \varphi)_*\mathcal{O}_{\tilde{X}} = (g \circ \pi)_*(\varphi_*\mathcal{O}_{\tilde{X}}) \\ &= \begin{cases} (g \circ \pi)_* \left(\bigoplus_{i=0}^{2} \mathcal{O}_P(-i) \otimes \pi^*(2iD')\right) & \text{if } (p,k) = (2,3) \\ (g \circ \pi)_* \left(\bigoplus_{i=0}^{1} \mathcal{O}_P(-2i) \otimes \pi^*(3iD')\right) & \text{if } (p,k) = (3,2) \end{cases} \\ &= \begin{cases} g_*(\pi_*\mathcal{O}_P) \oplus \; g_*(\pi_*\mathcal{O}_P(-1) \otimes \mathcal{O}_X(2D')) & \\ \qquad \oplus \, g_*(\pi_*\mathcal{O}_P(-2) \otimes \mathcal{O}_X(4D')) & \text{if } (p,k) = (2,3) \\ g_*(\pi_*\mathcal{O}_P) \oplus g_*(\pi_*\mathcal{O}_P(-2) \otimes \mathcal{O}_X(3D')) & \text{if } (p,k) = (3,2) \end{cases} \end{aligned}$$

Now since $\pi_*\mathcal{O}_P = \mathcal{O}_X$ and $\pi_*\mathcal{O}_P(-i) = 0$ for $i > 0$ we obtain $h_*\mathcal{O}_{\tilde{X}} = g_*\mathcal{O}_X$. □

Remark 14. Using another spectral sequence and 5-term exact sequence we can show the inclusion $H^0(C, R^1g_*(f_*\mathcal{O}_{\tilde{X}})) \subset H^0(C, R^1h_*\mathcal{O}_{\tilde{X}})$ but the equality does not hold in general.

Next we consider the construction (II), whose algorithm is as follows: Given a (pre-)Tango curve, we make a (pre-)Tango polarized surface (X, D) and a $\mathbb{P}^1$-bundle $\pi : P \longrightarrow X$ with the canonical section $F \subset P$ together with a purely inseparable cover $\pi|_G : G \longrightarrow X$ of degree p corresponding to D. Then choose $k = p^\ell r$ with $(p, r) = 1$ and $\ell \geq 1$ and let $\varphi : \tilde{X} \longrightarrow P$ be the normalization of P in $k(P)(R^{1/k})$ where $R \in K(P)$ is such that $(R) = G - (pF - p\pi^*(D))$.

Lemma 15. *Let (X, D), $D = kD'$ with $(2 \leq)\ k \in \mathbb{N}$, be a (pre-)Tango polarized surface by the Raynaud-Mukai construction. Then the construction (II) gives a Calabi-Yau threefold if and only if $(p, k, K) = (2, 4, 2D')$ or $(3, 3, D)$.*

Proof. Let (X, D) be a (pre-)Tango polarized surface by the Raynaud-Mukai construction. Then we obtain a $\mathbb{P}^1$-bundle $\pi : P \longrightarrow X$ together with the canonical section F and the purely inseparable cover $G \to X$ of degree p (see Theorem 4).

In Mukai construction, we take a kth cyclic cover of P where $(k, p) = 1$. This does not work as we have seen in Corollary 10. Thus we assume $(k, p) \neq 1$ and set $k = p^\ell r$ with $(p, r) = 1$, $\ell \geq 1$. Since we have $D = kD'$ and $G \sim pF - p\pi^*(D)$, there exists $R \in k(P)$ such that $(R) = G - pF + p\pi^*(kD')$. Now let $\varphi : \tilde{X} \longrightarrow P$ be the normalization of P in $k(P)(R^{1/k})$. Then if we set $\tilde{F} = (\varphi^*(F))_{red}$ and $\tilde{G} = (\varphi^*(G))_{red}$, we have $\varphi^*(G) = k\tilde{G}$ and $\varphi^*(F) = (k/p)\tilde{F}$ and $\tilde{G} \sim \tilde{F} - pf^*(D')$ where $f = \pi \circ \varphi$. Notice that we do not have the coefficient p for $\tilde{F}$ as in the case of $(p, k) = 1$. Now as in proof of Proposition 8, we compute

$$\begin{aligned}
\tilde{K} &\sim \varphi^* K_P + (k-1)\tilde{G} + (\frac{k}{p} - 1)\tilde{F} \\
&\sim \varphi^* K_P + \left(k + \frac{k}{p} - 2\right)\tilde{F} - p(k-1)f^*D' \\
&\sim (p^\ell r - p^{\ell-1} r - 2)\tilde{F} + f^*(K + (p^\ell r - p(p^\ell r - 1))D').
\end{aligned}$$

Then if $\tilde{X}$ is a Calabi-Yau threefold, i.e., $\tilde{K} \sim 0$, we must have $p^\ell r - p^{\ell-1} r - 2 = 0$ and $K + (p^\ell r - p(p^\ell r - 1))D' \sim 0$, from which we have $(\ell, r, p, k) = (1, 1, 3, 3)$ or $(2, 1, 2, 4)$ and

$$K \sim \begin{cases} 2D' & \text{if } (p, k) = (2, 4) \\ 3D'(= D) & \text{if } (p, k) = (3, 3) \end{cases}$$

$\square$

Now we can show

Proposition 16. *Calabi-Yau threefolds cannot be obtained by the construction (II).*

Proof. We assume that the (pre-)Tango polarized surface (X, D) is a fibration $f_1 : X \longrightarrow X_1$ over a Tango polarized curve (X_1, D_1) with $D_1 = k_1 D_1'$, which is a k_1th cyclic cover $\varphi_1 : X \longrightarrow P_1$ of a $\mathbb{P}^1$-fibration $\pi_1 : P_1 \longrightarrow X_1$ ramified over $F_1 + G_1 \subset P_1$ and we set $\tilde{F}_1 = (\varphi_1^*(F_1))_{red}$. In this situation, we have

$$K \sim (pk_1 - p - k_1 - 1)\tilde{F}_1 + f_1^*(K_{X_1} - (pk_1 - p - k_1)D_1')$$

by Proposition 8. We have $D = (k_1 - 1)\tilde{F}_1 + f_1^* D_1'$ by definition. Now we first consider the case $(p, k) = (2, 4)$. By Lemma 15 we have

$$2D' = \frac{1}{2}D = \frac{1}{2}(k_1 - 1)\tilde{F}_1 + \frac{1}{2}f_1^* D_1' \sim K = (k_1 - 3)\tilde{F}_1 + f_1^*(K_{X_1} - (k_1 - 2)D_1')$$

or otherwise

$$\frac{5 - k_1}{2}\tilde{F}_1 + f_1^*(\frac{2k_1 - 3}{2}D_1' - K_{X_1}) \sim 0,$$

which entails $k_1 = 5$ and $K_{X_1} = \frac{7}{2}D_1'$. But since D_1 is a (pre-)Tango structure we must have $\frac{7}{2}D_1' = K_{X_1} \geq pD_1 = 2 \cdot 4D_1' = 8D_1'$, a contradiction.

The case of $(p, k) = (3, 3)$ is similar. Since we must have $D \sim K$, we have $k_1 = 3$ and $K_{X_1} = 4D'$. But, since (X_1, D_1) is a Tango-Raynaud curve, we must have $4D_1' = K_{X_1} \geq pD_1 = 3k_1 D_1' = 9D_1'$, a contradiction. □

4. Cohomology of Calabi-Yau threefold with Tango-structure

In this section, we compute the cohomology $H^1(X, H^{-1})$ for arbitrary ample H under the assumption that X is a Calabi-Yau threefold on which Kodaira vanishing fails.

Theorem 17 (N. Shepherd-Barron [17])**.** *Let X be a normal locally complete intersection Fano threefold over the field k of* $\operatorname{char} k = p \geq 5$ *and L be an ample line bundle on X. Then we have $H^1(X, L^{-1}) = 0$.*

Recall that, for a polarized smooth variety (X, L), Kodaira non-vanishing $H^1(X, L^{-1}) \neq 0$ does not necessarily imply L is a (pre-)Tango structure. But by Enriques-Severi-Zariski's theorem, there exists $\ell > 0$ such that we have $H^1(X, L^{-p^{\ell+1}}) = 0$ but $H^1(X, L^{-p^{\ell}}) \neq 0$. Then such L^ℓ is at least a pre-Tango structure. Now based on these observations, we obtain

Theorem 18. *Let (X, L) be a smooth Calabi-Yau threefold over a field k of* $\operatorname{char} k = p \geq 5$ *with Kodaira non-vanishing $H^1(X, L^{-1}) \neq 0$. If L^ℓ is a Tango structure for some $\ell \geq 1$, then we have*

$$H^1(X, H^{-1}) = H^0(X, H^{-1} \otimes (\rho_* \mathcal{O}_Y / \mathcal{O}_X))$$

for every ample line bundle H on X, where $\rho : Y \longrightarrow X$ is a purely inseparable cover of degree p corresponding to the Tango structure as in Theorem 4.

Proof. By taking a sufficiently large power L^ℓ, $\ell \gg 0$, we can assume from the beginning that $H^1(X, L^{-p}) = 0$. Also, by the assumption we can assume that L is a Tango structure. Then by Theorem 4 we have a purely inseparable cover $\rho : Y \longrightarrow X$ of degree p and $\omega_Y \cong \rho^*(\omega_X \otimes L^{-p+1}) \cong (\rho^* L)^{-p+1}$, see II 6.1.6 [9]. Since ρ is a finite morphism and L is ample, $\rho^* L$ is also ample. Thus we know that Y is an integral Fano threefold. Also since L is a Tango structure, Y is smooth by Theorem 5. Now let H be an arbitrary ample line bundle on X. Then, since ρ is surjective, we have the following exact sequence

$$0 \longrightarrow H^{-1} \longrightarrow H^{-1} \otimes \rho_* \mathcal{O}_Y \longrightarrow H^{-1} \otimes \rho_* \mathcal{O}_Y / \mathcal{O}_X \longrightarrow 0,$$

from which we obtain the long exact sequence

$$H^0(X, \rho_*\rho^*H^{-1}) \longrightarrow H^0(X, H^{-1} \otimes \rho_*\mathcal{O}_Y/\mathcal{O}_X) \longrightarrow H^1(X, H^{-1}) \longrightarrow H^1(X, \rho_*\rho^*H^{-1}).$$

Now, we have $H^0(X, \rho_*\rho^*H^{-1}) = H^0(Y, \rho^*H^{-1}) = 0$ since ρ is finite and H is ample. Also $H^1(X, \rho_*\rho^*H^{-1}) = H^1(Y, \rho^*H^{-1})$ and this is 0 by Theorem 17. □

Recall that for a purely inseparable cover $p : Y \longrightarrow X$ of degree p there exists a p-closed rational vector field D on X such that $(\rho_*\mathcal{O}_Y)^D := \{f \in \rho_*\mathcal{O}_Y \ : \ D(f) = 0\} = \mathcal{O}_X$ (cf. [15]). Thus we have

Corollary 19. *Under the same assumption as Theorem 18, we have*

$$H^1(X, H^{-1}) = H^0(X, H^{-1} \otimes D(\rho_*\mathcal{O}_Y))$$

where D is a p-closed rational vector field on X corresponding to the purely inseparable cover ρ.

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References

[1] S. Cynk and D. van Straten. Small resolutions and non-liftable Calabi-Yau threefolds. Manuscripta Math. 130 (2009), no. 2, 233–249.

[2] P. Deligne, Relèvement des surfaces $K3$ en caractéristique nulle. Prepared for publication by Luc Illusie. Lecture Notes in Math., 868, Algebraic surfaces (Orsay, 1976–78), pp. 58–79, Springer, Berlin-New York, 1981

[3] P. Deligne and L. Illusie, Relevements modulo p^2 et décomposition du complexe de de Rham. Invent. Math. 89 (1987), no. 2, 247–270.

[4] T. Ekedahl, On non-liftable Calabi-Yau threefolds (preprint), math.AG/0306435

[5] H. Esnault and E. Viehweg, Lectures on vanishing theorems. DMV Seminar, 20. Birkhäuser Verlag, Basel, 1992.

[6] M. Hirokado, A non-liftable Calabi-Yau threefold in characteristic 3. Tohoku Math. J. (2) 51 (1999), no. 4, 479–487.

[7] M. Hirokado, H. Ito and N. Saito. Calabi-Yau threefolds arising from fiber products of rational quasi-elliptic surfaces. I. Ark. Mat. 45 (2007), no. 2, 279–296.

[8] M. Hirokado, H. Ito and N. Saito. Calabi-Yau threefolds arising from fiber products of rational quasi-elliptic surfaces. II. Manuscripta Math. 125 (2008), no. 3, 325–343

[9] J. Kollar, *Rational Curves on Algebraic Geometry*, Ergebnisse der Mathematik und ihre Grenzgebiete 3. Folge, Band 32, Springer, 1996.

[10] M. Maruyama, On a family of algebraic vector bundles, *Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki*, Kinokuniya, Tokyo 1973, pp95–146.

[11] S. Mukai, On counter-examples of Kodaira's vanishing theorem and Yau's inequality (in Japanese), Kinosaki Algebraic Geometry Symposium, 1979

[12] S. Mukai, Counterexamples of Kodaira's vanishing and Yau's inequality in higher dimensional variety of characteristic $p > 0$, RIMS preprint, 2005.

[13] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, 5. Oxford University Press, London, 1970 published for the Tata Institute of Fundamental Research, Bombay.

[14] M. Raynaud, M. Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. (French) C. P. Ramanujam—a tribute, pp. 273–278, Tata Inst. Fund. Res. Studies in Math., 8, Springer, Berlin-New York, 1978.

[15] P. Russell, Factoring Frobenius morphisms of an algebraic surface, Algebraic geometry, Bucharest 1982 (Bucharest, 1982), 366–380, Lecture Notes in Math., 1056, Springer, Berlin, 1984.

[16] S. Schröer, Some Calabi-Yau threefolds with obstructed deformations over the Witt vectors. Compos. Math. 140 (2004), no. 6, 1579–1592.

[17] N. I. Shepherd-Barron, Fano threefolds in positive characteristic. Compositio Math. 105 (1997), no. 3, 237–265.

[18] Y. Takayama, On non-vanishing of cohomologies of generalized Raynaud polarized surfaces. J. Pure Appl. Algebra 214 (2010), no. 7, 1110–1120.

[19] Y. Takeda. Pre-Tango structures and uniruled varieties. Colloq. Math. 108 (2007), no. 2, 193–216.

[20] H. Tango, On the behavior of cohomology classes of vector bundles with regard to Frobenius morphism (Japanese), Suurikaisekikenkyujo Kokyuroku, 1972.

[21] H. Tango, On the behavior of extensions of vector bundles under the Frobenius map, Nagoya Math. J., 48, 73–89, 1972.

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